

# A note on V-free 2-matchings

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## Abstract

Motivated by a conjecture of Liang [Y.-C. Liang. *Anti-magic labeling of graphs*. PhD thesis, National Sun Yat-sen University, 2013.], we introduce a restricted path packing problem in bipartite graphs that we call a V-free 2-matching. We verify the conjecture through a weakening of the hypergraph matching problem. We close the paper by showing that it is NP-complete to decide whether one of the color classes of a bipartite graph can be covered by a V-free 2-matching.

## 1 Introduction

Throughout the paper, graphs are assumed to be simple. Given an undirected graph  $G = (V, E)$  and a subset  $F \subseteq E$  of edges,  $F(v)$  denotes the set of edges in  $F$  incident to a node  $v \in V$ , and  $d_F(v) := |F(v)|$  is the **degree** of  $v$  in  $F$ . We say that  $F$  **covers** a subset of nodes  $X \subseteq V$  if  $d_F(v) \geq 1$  for every  $v \in X$ . Let  $b : V \rightarrow \mathbb{Z}_+$  be an upper bound function. A subset  $N \subseteq E$  of edges is called a **b-matching** if  $d_N(v)$  is at most  $b(v)$  for every node  $v \in V$ . For some integer  $t \geq 2$ , by a **t-matching** we mean a  $b$ -matching where  $b(v) = t$  for every  $v \in V$ . If  $t = 1$ , then a  $t$ -matching is simply called a **matching**.

A **hypergraph** is a pair  $H = (V, \mathcal{E})$  where  $V$  is a finite set of nodes and  $\mathcal{E}$  is a collection of subsets of  $V$ . The members of  $\mathcal{E}$  are called **hyperedges**, and for a hyperedge  $e \in \mathcal{E}$  let  $|e|$  denote its cardinality (as a subset of  $V$ ). In hypergraphs –unlike in graphs– we will allow hyperedges of cardinality 1 in this paper. A **matching** in a hypergraph is a collection of pairwise disjoint hyperedges, and the matching is said to be perfect if the union of the hyperedges in the matching contains every node. The **hypergraph matching problem** is to decide whether a given hypergraph has a perfect matching. Given a hypergraph  $H = (V, \mathcal{E})$ , we can represent it as a bipartite graph  $G_H = (U_V, U_{\mathcal{E}}; E)$ , where nodes of  $U_V$  correspond to nodes in  $V$ , nodes in  $U_{\mathcal{E}}$  correspond to hyperedges in  $\mathcal{E}$ , and there is an edge in  $G$  between a node  $u_v \in U_V$  (corresponding to  $v \in V$ ) and a node  $u_e \in U_{\mathcal{E}}$  (corresponding to  $e \in \mathcal{E}$ ) if and only if  $v \in e$  ( $G_H$  is also called the Levi graph of  $H$ ).

Let  $G = (S, T; E)$  be a bipartite graph. A path  $P = (\{u, v, w\}, \{uv, vw\})$  of length 2 with  $u, w \in S$  is called an **S-link**, and a **T-link** can be defined analogously. In [13], Liang proposed the following conjecture and showed that, if it is true, the conjecture implies that 4-regular graphs are antimagic (where a simple graph  $G = (V, E)$  is said to be **antimagic** if there exists a bijection  $f : E \rightarrow \{1, 2, \dots, |E|\}$  such that  $\sum_{e \in E(v_1)} f(e) \neq \sum_{e \in E(v_2)} f(e)$  for every pair  $v_1, v_2 \in V$ ).

**Conjecture 1.** *Assume that  $G = (S, T; E)$  is a bipartite graph such that each node in  $S$  has degree at most 4 and each node in  $T$  has degree at most 3. Then  $G$  has a matching  $M$  and a family  $\mathcal{F}$  of node-disjoint S-links such that every node  $v \in T$  of degree 3 is covered by an edge in  $M \cup (\cup_{P \in \mathcal{F}} P)$ .*

Observe that it suffices to verify the conjecture for the special case when each node in  $T$  has degree exactly 3, as we can simply delete nodes of degree less than 3. Although it was recently proved that regular graphs are antimagic [1], we prove the conjecture in Section 3 as it is interesting in its own. The proof is based on a weakening of the hypergraph matching problem.

While working on the proof of the conjecture, an interesting restricted path factor problem came to our attention. For simplicity, we will call a  $T$ -link a **V-path** (the name comes from the shape of these paths when  $T$  is placed ‘above’  $S$ , see Figure 1 for an illustration). It is easy to see that a 2-matching consists of pairwise node-disjoint paths and cycles. We call a 2-matching **V-free** if it does not contain a V-path as a connected component.

Consider the problem of finding a matching  $M$  and a family  $\mathcal{F}$  of node-disjoint S-links such that  $M \cup (\cup_{P \in \mathcal{F}} P)$  covers  $T$ . We can assume that  $M$  does not contain any edge of  $\bigcup \mathcal{F}$ , as such edges can be simply deleted from

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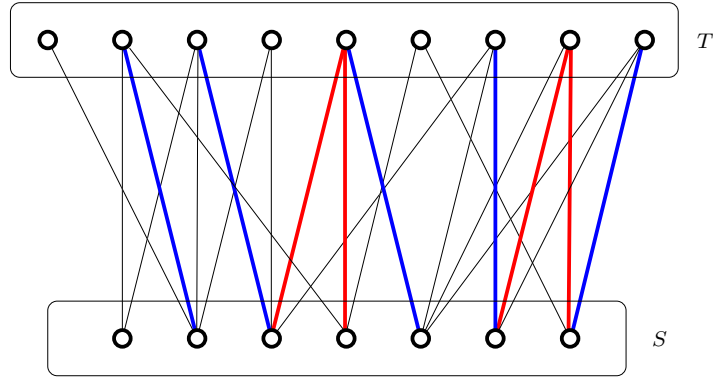


Figure 1: An illustration for Liang’s conjecture. Nodes in  $T$  have degree at most 3, and those in  $S$  have degree at most 4. The matching is highlighted with blue, the family of  $S$ -links is highlighted with red.

$M$ . Furthermore, we may assume that each node  $v \in T$  has degree at most 2 in  $M \cup (\cup_{P \in \mathcal{F}} P)$ . Indeed, if a node  $v \in T$  has degree 3 in  $M \cup (\cup_{P \in \mathcal{F}} P)$  then it is covered by both  $M$  and  $(\cup_{P \in \mathcal{F}} P)$ , so the edge in  $M$  incident to  $v$  can be deleted (see Figure 1). It is not difficult to see that  $M \cup (\cup_{P \in \mathcal{F}} P)$  is a  $\mathbf{V}$ -free 2-matching covering  $T$  in this case.

Conversely, given an arbitrary  $\mathbf{V}$ -free 2-matching  $N$  that covers  $T$ , edges can be left out from  $N$  in such a way that the resulting  $\mathbf{V}$ -free 2-matching  $N'$  still covers  $T$  and consists of paths of length 1 and 4, the latter having both end-nodes in  $T$ . Then  $N'$  can be partitioned into a matching and a family of node-disjoint  $S$ -links.

By the above, the problem of finding a matching  $M$  and a family  $\mathcal{F}$  of node-disjoint  $S$ -links whose union covers  $T$  is equivalent to finding a  $\mathbf{V}$ -free 2-matching  $N$  that covers  $T$ . The proof of Conjecture 1 shows that these problems can be solved when nodes in  $S$  have degree at most 4, and those in  $T$  have degree at most 3. However, in Section 4 we show that the problem of finding a  $\mathbf{V}$ -free 2-matching in a bipartite graph  $G = (S, T; E)$  covering  $T$  is NP-complete in general.

Let us now recall some well known results from matching theory that will be used below.

**Theorem 2.** *In a bipartite graph there exists a matching that covers every node of maximum degree.*

**Theorem 3** (Dulmage and Mendelsohn [3]). *Given a bipartite graph  $G = (S, T; E)$  and subsets  $X \subseteq S$ ,  $Y \subseteq T$ , if there exist two matchings  $M_X$  and  $M_Y$  in  $G$  such that  $M_X$  covers  $X$  and  $M_Y$  covers  $Y$  then there exists a matching  $M$  in  $G$  that covers  $X \cup Y$ .*

**Theorem 4** (Gallai-Edmonds Decomposition Theorem for graphs, see eg. [15]). *Given a graph  $G = (V, E)$ , let  $D$  be the set of nodes which are not covered by at least one maximum matching of  $G$ ,  $A$  be the set of neighbours of  $D$  and  $C := V - (D \cup A)$ . Then (a) the components of  $G[D]$  are factor-critical, (b)  $G[C]$  has a perfect matching, and (c)  $G$  has a matching covering  $A$ .*

The paper is organized as follows. Section 2 gives a brief overview of earlier results on restricted path packing problems. In Section 3, we introduce a variant of the hypergraph matching problem and prove a general theorem which in turn implies the conjecture. The paper is closed with a complexity result on  $\mathbf{V}$ -free 2-matchings in a bipartite graph  $G = (S, T, E)$  covering  $T$ , see Section 4.

## 2 Previous work

For a set  $\mathcal{F}$  of connected graphs, a spanning subgraph  $M$  of a graph  $G$  is called an  $\mathcal{F}$ -factor of  $G$  if every component of  $M$  is isomorphic to one of the members of  $\mathcal{F}$ . The **path** and **cycle** having  $n$  nodes are denoted by  $P_n$  and  $C_n$ , respectively. The **length** of  $P_n$  is  $n - 1$ , the number of its edges.

The problem of packing  $\mathcal{F}$ -factors is widely studied. Kaneko presented a Tutte-type characterization of graphs admitting a  $\{P_n | n \geq 3\}$ -factor [8]. Kano, Katona and Király [9] gave a simpler proof of Kaneko’s theorem and also a min-max formula for the maximum number of nodes that can be covered by a 2-matching not containing a single edge as a connected component. Such a 2-matching is often called **1-restricted**. These results were further generalized by Hartvigsen, Hell and Szabó [6] by introducing the so-called  **$k$ -piece packing** problem, where a  $k$ -piece is a connected graph with highest degree exactly  $k$ . In contrast with earlier approaches, their result is algorithmic, and so it provides a polynomial time algorithm for finding a 1-restricted 2-matching covering a maximum number of nodes. Later Janata, Loebl and Szabó [7] described a Gallai-Edmonds type

structure theorem for  $k$ -piece packings and proved that the node sets coverable by  $k$ -piece packings have a matroidal structure.

In [5], Hartvigsen considered the edge-max version of the 1-restricted 2-matching problem, that is, when a 1-restricted 2-matching containing a maximum number of edges is needed. He gave a min-max theorem characterizing the maximum number of edges in such a subgraph, and he also presented a polynomial algorithm for finding one. The notion of 1-restricted 2-matchings was generalized by Li [12] by introducing  **$j$ -restricted  $k$ -matchings** that are  $k$ -matchings with each connected component having at least  $j + 1$  edges. She considered the node-weighted version of the problem of finding a  $j$ -restricted  $k$ -matching in which the total weight of the nodes covered by the edges is maximal and presented a polynomial algorithm for the problem as well as a min-max theorem in the case of  $j < k$ . She also proved that the problem of maximizing the number of nodes covered by the edges in a  $j$ -restricted  $k$ -matching is NP-hard when  $j \geq k \geq 2$ .

A graph is called **cubic** if each node has degree 3. Cycle-factors and path-factors of cubic graphs are well-studied. The fundamental theorem of Petersen states that each 2-connected cubic graph has a  $\{C_n | n \geq 3\}$ -factor [14]. From Kaneko's theorem it follows that every connected cubic graph has a  $\{P_n | n \geq 3\}$ -factor. Kawarabayashi, Matsuda, Oda and Ota proved that every 2-connected cubic graph has a  $\{C_n | n \geq 4\}$ -factor, and if the graph has order at least six then it also has a  $\{P_n | n \geq 6\}$ -factor [11]. For bipartite graphs, these results were improved by Kano, Lee and Suzuki by showing that every connected cubic bipartite graph has a  $\{C_n | n \geq 6\}$ -factor, and if the graph has order at least eight then it also has a  $\{P_n | n \geq 8\}$ -factor [10].

Although the V-free 2-matching problem shows lots of similarities to these problems, it does not seem to fit in the framework of earlier approaches.

### 3 Extended matchings

While working on Conjecture 1, we arrived at a relaxation of the hypergraph matching problem that we call the **extended matching problem**. An **extended matching** of a hypergraph  $H = (V, \mathcal{E})$  is a disjoint collection of hyperedges and pairs of nodes where a pair  $(u, v)$  may be used only if there exists a hyperedge  $e \in \mathcal{E}$  with  $u, v \in e$ . An extended matching is **perfect** if it covers the node-set of  $H$ . Note that one can decide in polynomial time if a hypergraph has a perfect extended matching by the results of [2] (see also Theorem 4.2.16 in [16]). Indeed, given a hypergraph  $H = (V, \mathcal{E})$ , consider its bipartite representation  $G_H = (U_V, U_{\mathcal{E}}; E)$ . Then a perfect extended matching in  $H$  corresponds to a subgraph in  $G_H$  in which nodes of  $U_V$  have degree one, and a node  $u_e \in U_{\mathcal{E}}$  corresponding to  $e \in \mathcal{E}$  has degree  $|e|$ , or any even number not greater than  $|e|$ .

However, we have found a simple proof of the following result, a special case of the extended matching problem, which implies Conjecture 1, as we show below.

**Theorem 5.** *In a 3-uniform hypergraph  $H = (V, \mathcal{E})$  there exists an extended matching that covers the nodes of maximum degree in  $H$ .*

Theorem 5 is the special case of a more general result (Corollary 9) that we introduce below. Before doing so, we show that Theorem 5 implies Conjecture 1.

*Proof of Conjecture 1.* Recall that it suffices to verify the conjecture for graphs  $G = (S, T; E)$  with  $d_E(v) = 3$  for every  $v \in T$ . Such a  $G$  is the incidence graph (or Levi graph) of a 3-uniform hypergraph  $H = (S, \mathcal{E})$  in which each node has degree at most 4.

Let  $S' \subseteq S$  denote the set of nodes having degree 4 in  $H$ . By Theorem 5,  $H$  has an extended matching covering  $S'$ . That is,  $S'$  can be covered by pairwise node-disjoint  $S$ -links and  $S$ -claws of  $G$ , where an  $S$ -claw is a star with 3 edges having its center node in  $T$ . We denote the edge-set of these  $S$ -links and claws by  $N$ .

Let  $T'$  be the set of nodes in  $T$  not covered by  $N$ . As  $d_{E-N}(v) \leq 3$  for each  $v \in S$ ,  $T'$  can be covered by a matching  $M$  disjoint from  $N$ , by Theorem 2. By leaving out an edge from each  $S$ -claw of  $N$ , we get a matching  $M$  and a family of  $S$ -links whose union together covers  $T$ .  $\square$

Let us now introduce and prove a generalization of Theorem 5. We call a hypergraph  $H = (V, \mathcal{E})$  **oddly uniform** if every hyperedge has odd cardinality. The **quasi-degree** of a node  $v \in V$  is defined as  $d^-(v) := \sum[|e| - 1 : v \in e \in \mathcal{E}]$ , and the hypergraph is  **$\Delta$ -quasi-regular** (or **quasi-regular** for short) if  $d^-(v) = \Delta$  for each  $v \in V$  where  $\Delta \in \mathbb{Z}_+$ . Note that a uniform regular hypergraph is quasi-regular.

**Theorem 6.** *Every oddly uniform quasi-regular hypergraph has a perfect extended matching.*

*Proof.* Assume that  $H = (V, \mathcal{E})$  is an oddly uniform  $\Delta$ -quasi-regular hypergraph, and let  $G = (V, E)$  denote the graph obtained by replacing each hyperedge  $e \in \mathcal{E}$  with a complete graph on node-set  $e \subseteq V$ . That is, there are as many parallel edges between  $u$  and  $v$  in  $E$  as the number of hyperedges containing both  $u$  and  $v$ . Note that the quasi-regularity of  $H$  is equivalent to the regularity of  $G$ .

If  $G$  admits a perfect matching  $M$ , then  $M$  is a perfect extended matching of  $H$  and we are done.

Assume that  $G$  does not have a perfect matching. Take the Gallai-Edmonds decomposition of  $G$  into sets  $D$ ,  $A$  and  $C$  (see Theorem 4). Let  $D_1$  be the union of those connected components of  $G[D]$  that span a hyperedge  $e \in \mathcal{E}$  in  $H$ , and  $D_2 := D - D_1$ .

**Claim 7.** *Every component  $K$  of  $G[D_1]$  has a perfect extended matching in  $H$ .*

*Proof.* As  $K$  is factor-critical, it has a perfect matching after deleting the nodes of any of its odd cycles (including the case when the cycle consists of a single node). Let  $e \in \mathcal{E}$  be a hyperedge spanned by  $K$ . By the above,  $G[K - e]$  has a perfect matching, which together with  $e$  form a perfect extended matching of  $K$ , proving the claim.  $\square$

**Claim 8.** *For every component  $K$  of  $G[D_2]$  we have  $d_G(K) \geq \Delta$ .*

*Proof.* Let  $u \in K$  be an arbitrary node.  $K$  does not span a hyperedge in  $H$ , hence for every hyperedge  $e$  containing  $u$  we have  $e \cap K \neq \emptyset$ ,  $e \cap A \neq \emptyset$  and  $e \subseteq K \cup A$ . By the definition of  $G$ , there are at least  $\sum[|e \cap K| \cdot |e \cap A| : u \in e \in \mathcal{E}] \geq \sum[|e| - 1 : u \in e \in \mathcal{E}] = \Delta$  edges between  $K$  and  $A$ , thus concluding the proof of the claim.  $\square$

Let  $G' = (D', A; F)$  denote the bipartite graph obtained from  $G$  by deleting the nodes of  $C$  and the edges induced by  $A$ , and by contracting each component of  $G[D]$  to a single node (the set of new nodes is denoted by  $D'$ ). Nodes of  $D'$  are partitioned into sets  $D'_1$  and  $D'_2$  accordingly. As  $d_{G'}(v) \leq \Delta$  for each  $v \in A$ , Claim 8 and Theorem 2 imply that  $G'$  has a matching covering  $D'_2$ . By Theorem 4 (c),  $G'$  has a matching covering  $A$ , hence the result of Dulmage and Mendelsohn (Theorem 3) implies that  $G'$  has a matching  $M'$  covering  $A$  and  $D'_2$  simultaneously. Considering  $M'$  as a matching in  $G$  and using Theorem 4 (a) and (b),  $M'$  can be extended to a matching  $M$  of  $G$  that covers every node that is in  $C \cup A$  or in a component of  $G[D]$  that is incident to an edge in  $M'$ . By Claim 7, there is an extended matching covering the nodes of the remaining components of  $G[D]$ , since they fall in  $D_1$ . The union of  $M$  and this extended matching forms a perfect extended matching of  $H$ . This completes the proof of the theorem.  $\square$

As a consequence, we get the following result.

**Corollary 9.** *Every oddly uniform hypergraph has an extended matching that covers the set of nodes having maximum quasi-degree.*

*Proof.* Let  $H = (S, \mathcal{E})$  be an oddly uniform hypergraph and let  $\Delta$  denote the maximum quasi-degree in  $H$ . The **deficiency** of a node  $v \in S$  is  $\gamma(v) := \Delta - d^-(v)$ . A node  $v \in S$  is called **deficient** if  $\gamma(v) > 0$ . As  $H$  is oddly uniform,  $\gamma(v)$  is even for every node  $v$ .

It suffices to show that  $H$  can be extended to a  $\Delta$ -quasi-uniform hypergraph  $H' = (V', \mathcal{E}')$  by adding further nodes and hyperedges. Indeed, by Theorem 6,  $H'$  admits a perfect extended matching whose restriction to the original hypergraph gives an extended matching covering each node having quasi-degree  $\Delta$ .

If there is no deficient node in  $H$ , then we are done. Otherwise consider the hypergraph obtained by taking the disjoint union of three copies of  $H$ , denoted by  $H_1$ ,  $H_2$  and  $H_3$ , respectively. For each deficient node  $v \in S$ , add  $\gamma(v)$  copies of the hyperedge  $\{v_1, v_2, v_3\}$  to the hypergraph, where  $v_i$  denotes the copy of  $v$  in  $H_i$ . The hypergraph  $H'$  thus obtained is clearly  $\Delta$ -quasi-regular.  $\square$

## 4 Complexity result

In what follows we show that deciding the existence of a  $\mathbf{V}$ -free 2-matching covering  $T$  is NP-complete in general. We will use reduction from the following problem (see [4, (SP2)]).

**Theorem 10** (3-dimensional matching). *Let  $H = (X, Y, Z; \mathcal{E})$  be a tripartite 3-regular 3-uniform hypergraph, meaning that each node  $v \in X \cup Y \cup Z$  is contained in exactly 3 hyperedges, and each hyperedge  $e \in \mathcal{E}$  contains exactly one node from all of  $X, Y$  and  $Z$ . It is NP-complete to decide whether  $H$  has a perfect matching, that is, a 1-regular sub-hypergraph.*

Our proof is inspired by the construction of Li for proving the NP-hardness of maximizing the number of nodes covered by the edges in a 2-restricted 2-matching [12].

**Theorem 11.** *Given a bipartite graph  $G = (S, T; E)$  with maximum degree 4, it is NP-complete to decide whether  $G$  has a  $\mathbf{V}$ -free 2-matching covering  $T$ .*

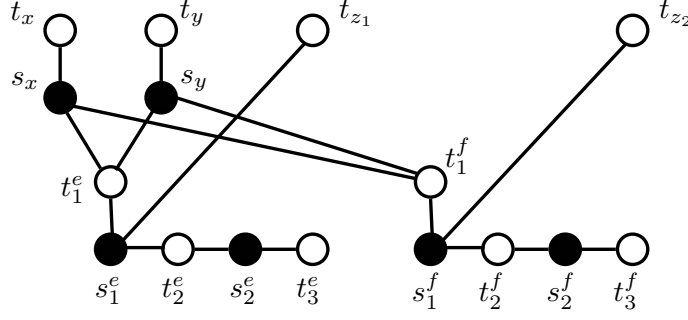


Figure 2: Gadgets corresponding to hyperedges  $e = \{x, y, z_1\}$  and  $f = \{x, y, z_2\}$

*Proof.* We prove the theorem by reduction from the 3-dimensional matching problem. Take a 3-uniform 3-regular tripartite hypergraph  $H = (X, Y, Z; \mathcal{E})$ . For a hyperedge  $e \in \mathcal{E}$ , we use the following notions:  $x_e := e \cap X$ ,  $y_e := e \cap Y$  and  $z_e := e \cap Z$ .

We construct an undirected bipartite graph as follows. For each node  $x \in X$  and  $y \in Y$ , add a pair of nodes  $s_x, t_x$  and  $s_y, t_y$  to  $G$ , respectively, with  $s_x, s_y \in S$  and  $t_x, t_y \in T$ . For each node  $z \in Z$ , add a single node  $t_z$  to  $T$ . Furthermore, for each  $x \in X$  and  $y \in Y$  add the edges  $s_x t_x$  and  $s_y t_y$  to  $E$ .

We assign a path  $P_e$  with node set  $V(P_e) = \{t_1^e, s_1^e, t_2^e, s_2^e, t_3^e\}$  and edge set  $E(P_e) = \{t_1^e s_1^e, s_1^e t_2^e, t_2^e s_2^e, s_2^e t_3^e, t_3^e\}$  of length four to each hyperedge  $e \in \mathcal{E}$  and add edges  $s_{x_e} t_1^e$ ,  $s_{y_e} t_1^e$  and  $t_{z_e} s_1^e$  to  $E$  (see Figure 2). It is easy to check that the graph thus arising is bipartite and has maximum degree 4 (here we use that every node  $v \in X \cup Y \cup Z$  is contained in exactly 3 hyperedges of  $H$ ).

We claim that  $H$  admits a perfect matching if and only if  $G$  has a  $V$ -free 2-matching covering  $T$ , which proves the theorem. Assume first that  $H$  has a perfect matching and let  $\mathcal{M} \subseteq \mathcal{E}$  be the set of matching hyperedges. Then

$$M := \bigcup_{e \in \mathcal{M}} \{s_{x_e} t_{x_e}, s_{y_e} t_{y_e}, s_{x_e} t_1^e, s_{y_e} t_1^e, t_{z_e} s_1^e, E(P_e) - t_1^e s_1^e\} \cup \bigcup_{e \notin \mathcal{M}} E(P_e)$$

is a  $V$ -free 2-matching covering  $T$  (see Figure 3).

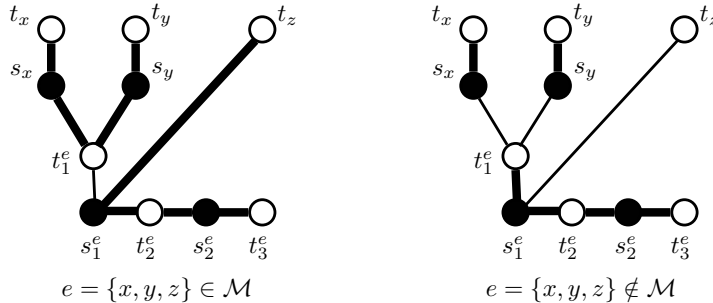


Figure 3: Edges included in  $M$  depending on whether  $e \in \mathcal{M}$  or not

For the other direction, take a  $V$ -free 2-matching  $M$  of  $G$  covering  $T$ . Observe that  $s_x t_x, s_y t_y \in M$  for each  $x \in X$  and  $y \in Y$  as  $M$  covers  $T$ . Moreover,  $M$  is  $V$ -free hence  $t_1^e s_1^e \notin M$  implies  $s_{x_e} t_1^e, s_{y_e} t_1^e \in M$ . We may assume that  $E(P_e) - t_1^e s_1^e \subseteq M$  for each  $e \in \mathcal{E}$ . Indeed,  $M$  has to cover  $t_2^e$  and  $t_3^e$ , hence the  $V$ -freeness of  $M$  implies  $s_1^e t_2^e, s_2^e t_3^e \in M$ . Consequently,  $t_2^e s_2^e \in M$  can be assumed.

We claim that  $d_M(t_z) = 1$  for each  $z \in Z$ . Indeed, if  $t_{z_e} s_1^e \in M$  for some  $e \in \mathcal{E}$  then  $s_{x_e} t_1^e, s_{y_e} t_1^e \in M$ . In other words, if  $t_{z_e} s_1^e \in M$  then  $e$  ‘reserves’ nodes  $s_{x_e}, s_{y_e}$  and  $t_{z_e}$  for  $M$  being a  $V$ -free 2-matching. On the other hand, for each  $x \in X$  there is at most one  $e \in \mathcal{E}$  such that  $s_{x_e} t_1^e \in M$ , and the same holds for each  $y \in Y$ . As the hypergraph is 3-uniform and 3-regular, we have  $|X| = |Y| = |Z|$ . Hence the number of edges of form  $t_{z_e} s_2^e$  in  $M$  can not exceed the cardinality of these sets. Let

$$\mathcal{M} := \{e \in \mathcal{E} : t_{z_e} s_2^e \in M\}.$$

By the above,  $\mathcal{M}$  is a 1-regular subhypergraph, thus concluding the proof.  $\square$

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